

Disformal invariance of curvature perturbation

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We show that under a general disformal transformation the linear comoving curvature perturbation is not identically invariant, but is invariant on superhorizon scales for any theory that is disformally related to Horndeski's theory. The difference between disformally related curvature perturbations is found to be given in terms of the comoving density perturbation associated with a single canonical scalar field. In General Relativity it is well-known that this quantity vanishes on superhorizon scales through the Poisson equation that is obtained on combining the Hamiltonian and momentum constraints, and we confirm that a similar result holds for any theory that is disformally related to Horndeski's scalar-tensor theory so long as the invertibility condition for the disformal transformation is satisfied. We also consider the curvature perturbation at full nonlinear order in the unitary gauge, and find that it is invariant under a general disformal transformation if we assume that an attractor regime has been reached. Finally, we also discuss the counting of degrees of freedom in theories disformally related to Horndeski's.

I. INTRODUCTION

A primordial epoch of inflation and the late-time accelerated expansion of the universe constitute two key elements of the standard model of modern cosmology, which is in very good agreement with observational data. Many of the models proposed to try and explain these two epochs of accelerated expansion rely on the introduction of an additional scalar degree of freedom, either in the form of an unknown scalar field in the matter sector, such as an inflaton or quintessence field, or as part of a modified gravity sector, such as in $f(R)$ gravity or Brans-Dicke scalar-tensor gravity.

Recently, efforts have been made to determine the most general form of scalar-tensor theory that encompasses the examples mentioned above and more. In the spirit of effective field theories, such a theory would allow one to introduce a common parameterisation for a wide range of models, making it much easier to understand the relation between different models and to compare their predictions with observations. In trying to construct the most general form of scalar-tensor action, a key requirement is that the resulting equations of motion are second order in time derivatives. The appearance of higher order derivatives is generally associated with the presence of additional, ghost-like degrees of freedom, and the associated Hamiltonian becomes unbounded from below, both in the presence of even higher order derivative terms [1] and odd higher order derivative terms [2]. As a result, if the system is coupled to another “normal” system, then the total system will develop a so-called Ostrogradsky instability. The most general form of scalar-tensor action that gives rise to second order equations of motion was derived by Horndeski over 40 years ago [3], and was rederived just a few years ago in the context of so-called Galileon models [4–7]. More recently, however, it has become apparent that there exist theories that do not belong to Horndeski's theory but that nevertheless do not suffer from Ostrogradsky instabilities, propagating only 3 degrees of freedom [8–15]. Whilst these theories may appear to give rise to higher-order equations of motion at the level of the Euler-Lagrange equations – which is why they are not included in Horndeski's theory – it has been shown that the higher-order time derivatives can be removed by making use of the time derivative of a special linear combination of the gravitational equations of motion, thus rendering the equations of motion second order with respect to time derivatives [16]. These theories are therefore interesting and their phenomenology has been investigated in the literature, e.g. the screening mechanism [17–19] and possible observational signatures [20].

In exploring this class of general scalar-tensor theories, use is often made of disformal transformations of the metric, which take the form [21]

$$\tilde{g}_{\mu\nu} = \alpha(\phi, X)g_{\mu\nu} + \beta(\phi, X)\partial_\mu\phi\partial_\nu\phi, \quad (1.1)$$

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where $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$. This is a generalisation of the more familiar conformal transformations, for which $\beta(\phi, X) = 0$ and $\alpha(\phi, X) \rightarrow \alpha(\phi)$. The different representations of a theory, written in terms of disformally related metrics, are often referred to as being written in different “frames”. In some cases a transformation of the form (1.1) can be used to remove non-minimal coupling between the scalar field ϕ and the Ricci scalar or Einstein tensor at the level of the action, leaving only a canonical Einstein-Hilbert term [8, 22, 23]. This particular frame, if it exists, is referred to as the Einstein frame. It is known that the form of Horndeski’s action is preserved under disformal transformations if α and β only depend on ϕ [8, 23]. Similarly, the form of the theories beyond Horndeski’s considered in [9, 12] is known to be preserved under disformal transformations with $\alpha = \alpha(\phi)$ and $\beta = \beta(\phi, X)$. Allowing for an X -dependence of α , however, allows one to transform between theories belonging to the class considered in [9, 12] and those that lie outside it [8, 12].

In many cases it is easier to solve for the dynamics of the scalar field ϕ coupled to gravity if we first rewrite the theory in terms of a metric that is disformally related to the original metric as in (1.1). Having solved for ϕ and $\tilde{g}_{\mu\nu}$, however, it is often the case that we would like to relate these quantities back to the original metric $g_{\mu\nu}$, for example if matter is minimally coupled to this metric, i.e. it defines the so-called Jordan frame. In the context of cosmology we are particularly interested in the transformation properties of perturbations, and especially the so-called comoving curvature perturbation, \mathcal{R}_c , which is defined as the curvature perturbation on time-slices of constant ϕ . It has been known for some time that the comoving curvature perturbation is invariant under conformal transformations with $\beta(\phi, X) = 0$ and $\alpha(\phi, X) \rightarrow \alpha(\phi)$, both at the linear level [24] and the fully nonlinear level [25, 26]. More recently, the invariance of \mathcal{R}_c was also confirmed for transformations where $\alpha(\phi, X) \rightarrow \alpha(\phi)$ and $\beta(\phi, X) \rightarrow \beta(\phi)$ [27], and finally for the case where $\alpha(\phi, X) \rightarrow \alpha(\phi)$ and β has both ϕ - and X -dependence [28]. The disformal invariance of the comoving curvature perturbation is a very useful result, as it means that if we are ultimately only interested in the comoving curvature perturbation then we are free to solve the system in whichever frame is most convenient. It is thus natural to ask whether or not the disformal invariance of \mathcal{R}_c holds for the most general form of disformal transformation, where we allow for an X -dependence of both α and β , and this is the question we address in this paper. We will show that the case where an X -dependence of α is included is crucially different to the previously considered cases, and the comoving curvature perturbation is not identically invariant under such a disformal transformation.

The paper is organised as follows. In §II, we investigate the transformation properties of linear perturbations, choosing to leave the gauge unfixed and to work with gauge-invariant quantities. We elucidate that the comoving curvature perturbation is not identically invariant under disformal transformations when one allows for an X -dependence of α , but that the difference is given in terms of the gauge-invariant comoving density perturbation associated with a single canonical scalar field. We then show that in the context of Horndeski’s theory a sufficient condition for the comoving density perturbation to vanish is $\tilde{\mathcal{R}}_c = 0$, and that under reasonable assumptions – i.e. neglecting the so-called decaying mode of \mathcal{R}_c – this condition is satisfied on super-horizon scales. Details of this calculation are presented in Appendix A. Consequently, we conclude that on superhorizon scales, and under reasonable assumptions, the comoving curvature perturbation is disformally invariant for any theory that is disformally related to Horndeski’s theory. In §III, we consider the transformation properties of perturbations at the nonlinear level, and in this analysis we find it easier to make use of the unitary gauge, where $\delta\phi = 0$, which makes β irrelevant to the transformation law of the comoving curvature perturbation. We find that the comoving curvature perturbation is invariant under disformal transformations if we assume that an attractor regime has been reached. In such an attractor regime X can be re-expressed as a function of ϕ , meaning that the situation is exactly the same as in the case of an X -independent α . Using our result, we deduce that in the attractor regime the nonlinear curvature perturbation is conserved on superhorizon scales in any theory that is related to Horndeski’s by a general disformal transformation. In §IV, we discuss the counting of degrees of freedom in theories disformally related to Horndeski’s. Given that we know Horndeski’s theory to be healthy, i.e. its equations of motion are second order and it propagates only 3 degrees of freedom, we focus our attention on how the counting of degrees of freedom is affected by a disformal transformation. Using a toy model we demonstrate that, as one would naively expect, the number of degrees of freedom as determined by a full Hamiltonian analysis should be unaffected by a disformal transformation, which is agreement with the recent analysis in [29]. §V is devoted to conclusions.

II. LINEAR ANALYSIS

We consider the disformal transformation given in (1.1). To make it a well-defined redefinition of fields, we require that the inverse disformal transformation from the tilded frame to un-tilded frame also exists, which requires the existences of $\tilde{g}^{\mu\nu}$ and the solvability of $X = X(\phi, \tilde{X})$.

Firstly, from (1.1) we can determine that the inverse metric $\tilde{g}^{\mu\nu}$ is given as

$$\tilde{g}^{\mu\nu} = \frac{g^{\mu\nu}}{\alpha(\phi, X)} - \frac{\beta(\phi, X)\partial^\mu\phi\partial^\nu\phi}{\alpha(\phi, X)[\alpha(\phi, X) - 2X\beta(\phi, X)]}, \quad (2.1)$$

where $\partial^\mu \phi = g^{\mu\nu} \partial_\nu \phi$. From this expression we see that the inverse $\tilde{g}^{\mu\nu}$ will exist as long as

$$\alpha(\alpha - 2X\beta) \neq 0. \quad (2.2)$$

This could also have been inferred from the relation between the determinants of $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$, which can be derived by contracting (1.1) with $g^{\mu\nu}$ and taking the determinant [30]

$$g^{\mu\nu} \tilde{g}_{\nu\alpha} = \alpha \left(\delta^\mu_\alpha + \frac{\beta}{\alpha} \partial^\mu \phi \partial_\alpha \phi \right) \Rightarrow \frac{\tilde{g}}{g} = \alpha^3 (\alpha - 2X\beta), \quad (2.3)$$

where \tilde{g} and g are the determinants of $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ respectively.

Secondly, let us consider the solvability condition for X in terms of ϕ and \tilde{X} . The complication that arises in trying to invert the transformation given in (1.1) is the appearance of $g^{\mu\nu}$ in X . As discussed in [31], in order to be able to invert the transformation we thus need to express $X = -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / 2$ in terms of $\tilde{X} = -\tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / 2$. When (2.2) is satisfied, we can contract (2.1) with $\partial_\mu \phi \partial_\nu \phi$ and obtain the relation

$$\tilde{X} = \frac{X}{\alpha(\phi, X) - 2X\beta(\phi, X)}. \quad (2.4)$$

The solubility of this relation for X requires $\partial \tilde{X} / \partial X \neq 0$, namely,

$$\frac{\alpha - X\alpha_X + 2X^2\beta_X}{(\alpha - 2X\beta)^2} \neq 0, \quad (2.5)$$

where $\alpha_X = \partial_X \alpha$ and $\beta_X = \partial_X \beta$.^{*1} The same condition can also be determined by considering the Jacobian $\partial \tilde{g}_{\mu\nu} / \partial g_{\alpha\beta}$ and requiring its determinant to be non-vanishing, as in [8]. Combined with (2.2),

$$\alpha(\alpha - 2X\beta)(\alpha - X\alpha_X + 2X^2\beta_X) \neq 0 \quad (2.6)$$

is the necessary and sufficient condition for the invertibility of the disformal transformation. If (2.6) is satisfied, we can solve (2.4) for $X = X(\phi, \tilde{X})$ and obtain the inverse disformal transformation

$$g_{\mu\nu} = \frac{1}{\alpha(\phi, X(\phi, \tilde{X}))} \tilde{g}_{\mu\nu} - \frac{\beta(\phi, X(\phi, \tilde{X}))}{\alpha(\phi, X(\phi, \tilde{X}))} \partial_\mu \phi \partial_\nu \phi. \quad (2.7)$$

In the following, we consider disformal transformations that satisfy the invertibility condition (2.6).

We first focus on a linear analysis of perturbations. Let us take a line element of the form:

$$\begin{aligned} ds^2 &\equiv g_{\mu\nu} dx^\mu dx^\nu \\ &= -(1 + 2A) dt^2 + 2a(\partial_i B - S_i) dx^i dt \\ &\quad + a^2[(1 - 2\psi)\delta_{ij} + 2\partial_i \partial_j E + \partial_i F_j + \partial_j F_i + h_{ij}] dx^i dx^j, \end{aligned} \quad (2.8)$$

where F_i and h_{ij} satisfy $F_{i,i} = h_{ii} = h_{ij,i} = 0$.^{*2} We then consider a disformal transformation of the form (1.1).

$$\begin{aligned} d\tilde{s}^2 &\equiv \tilde{g}_{\mu\nu} dx^\mu dx^\nu \\ &= [-\alpha_0(1 + 2A) - \delta\alpha + (\beta_0 + \delta\beta)\dot{\phi}^2 + 2\beta_0\dot{\phi}\delta\phi] dt^2 \\ &\quad + 2[\alpha_0 a(\partial_i B - S_i) + \beta_0 \dot{\phi} \partial_i \delta\phi] dt dx^i \\ &\quad + \alpha_0 a^2 \left[\left(1 - 2\psi + \frac{\delta\alpha}{\alpha_0} \right) \delta_{ij} + 2\partial_i \partial_j E + \partial_i F_j + \partial_j F_i + h_{ij} \right] dx^i dx^j, \end{aligned} \quad (2.9)$$

where a dot denotes d/dt and we have decomposed $\alpha = \alpha_0 + \delta\alpha$ and $\beta = \beta_0 + \delta\beta$. Note that we are choosing to leave the gauge unfixed, which will make the interpretation later on more obvious. We would then like to rewrite this new line element in the same form as (2.8), namely as

$$\begin{aligned} d\tilde{s}^2 &= -(1 + 2\tilde{A}) d\tilde{t}^2 + 2\tilde{a}(\partial_i \tilde{B} - \tilde{S}_i) dx^i d\tilde{t} \\ &\quad + \tilde{a}^2[(1 - 2\tilde{\psi})\delta_{ij} + 2\partial_i \partial_j \tilde{E} + \partial_i \tilde{F}_j + \partial_j \tilde{F}_i + \tilde{h}_{ij}] dx^i dx^j. \end{aligned} \quad (2.10)$$

^{*1} We will adopt a similar notation for derivatives with respect to ϕ , e.g. $\alpha_\phi = \partial_\phi \alpha$.

^{*2} Note that repeated indices are summed over.

At background level this then gives us

$$\tilde{a} = a\sqrt{\alpha_0} \quad \text{and} \quad d\tilde{t} = \sqrt{\alpha_0 - \beta_0\dot{\phi}^2} dt, \quad (2.11)$$

where we note that a dot still corresponds to taking the derivative with respect to t rather than \tilde{t} . At the level of perturbations, from the 00-component we obtain

$$\tilde{A} = \frac{1}{\alpha_0 - \beta_0\dot{\phi}^2} \left(\alpha_0 A + \frac{1}{2} \delta\alpha - \frac{1}{2} \dot{\phi}^2 \delta\beta - \beta_0 \dot{\phi} \delta\dot{\phi} \right). \quad (2.12)$$

From the 0*i*-component we have

$$\tilde{B} = \sqrt{\frac{\alpha_0}{\alpha_0 - \beta_0\dot{\phi}^2}} \left(B + \frac{\beta_0 \dot{\phi}}{\alpha_0 a} \delta\phi \right) \quad \text{and} \quad \tilde{S}_i = S_i. \quad (2.13)$$

Finally, from the *ij*-component we find

$$\tilde{\psi} = \psi - \frac{\delta\alpha}{2\alpha_0}, \quad (2.14)$$

whilst E , F_i , and h_{ij} remain unchanged. As such, we see that the vector and tensor perturbation are invariant under a general disformal transformation at linear level. Whilst the above expressions are problematic if $\alpha_0 - \beta_0\dot{\phi}^2$ vanishes, recall that we are assuming $\alpha - 2X\beta \neq 0$ for the existence of $\tilde{g}^{\mu\nu}$, which yields $\alpha_0 - \beta_0\dot{\phi}^2 \neq 0$ at background level.^{*3}

Turning to the gauge-invariant comoving curvature perturbation, \mathcal{R}_c , it is defined in the original frame as

$$\mathcal{R}_c = -\psi - \frac{H}{\dot{\phi}} \delta\phi, \quad (2.15)$$

where $H = \dot{a}/a$. In the new frame we similarly have

$$\tilde{\mathcal{R}}_c = -\tilde{\psi} - \frac{\tilde{H}}{d\phi/d\tilde{t}} \delta\phi, \quad (2.16)$$

where $\tilde{H} = (1/\tilde{a})d\tilde{a}/d\tilde{t}$. Using the background relations (2.11) we have

$$\frac{d\phi}{d\tilde{t}} = \frac{\dot{\phi}}{\sqrt{\alpha_0 - \beta_0\dot{\phi}^2}} \quad \text{and} \quad \tilde{H} = \frac{1}{\sqrt{\alpha_0 - \beta_0\dot{\phi}^2}} \left(H + \frac{\dot{\alpha}_0}{2\alpha_0} \right), \quad (2.17)$$

which, on combining with (2.14), gives us

$$\tilde{\mathcal{R}}_c = \mathcal{R}_c + \frac{1}{2\alpha_0} \left(\delta\alpha - \frac{\dot{\alpha}_0}{\dot{\phi}} \delta\phi \right). \quad (2.18)$$

In the case that $\alpha = \alpha(\phi)$ we have $\delta\alpha - \dot{\alpha}_0 \delta\phi / \dot{\phi} = 0$, meaning that $\mathcal{R}_c = \tilde{\mathcal{R}}_c$, which is consistent with [27, 28]. However, in the case that we allow for an X -dependence of α , we more generally get

$$\delta\alpha - \frac{\dot{\alpha}_0}{\dot{\phi}} \delta\phi = \alpha_{0X} \left(\delta X - \frac{\dot{X}_0}{\dot{\phi}} \delta\phi \right), \quad (2.19)$$

where $\alpha_{0X} = \partial\alpha_0/\partial X$ and $X_0 = \dot{\phi}^2/2$. Using $\delta X = \dot{\phi}(\delta\phi - \dot{\phi}A)$, we thus find

$$\tilde{\mathcal{R}}_c - \mathcal{R}_c = \frac{\alpha_{0X}}{2\alpha_0} \epsilon_s, \quad (2.20)$$

^{*3} In fact, the requirement that the perturbative expansion be valid in the disformally related frame, e.g. $\tilde{A} \ll 1$, will put tighter constraints on the background-dependent coefficients appearing in eqs. (2.12)–(2.14) than those imposed by the invertibility of the transformation, but we will not consider this issue here.

where

$$\epsilon_s \equiv \dot{\phi}(\dot{\delta\phi} - \dot{\phi}A) - \ddot{\phi}\delta\phi = \delta X - \ddot{\phi}\delta\phi. \quad (2.21)$$

We thus see that the comoving curvature perturbation is not identically invariant under disformal transformations with $\alpha = \alpha(\phi, X)$. Note that if one takes the gauge $\delta\phi = 0$, $\tilde{\mathcal{R}}_c$ as determined by (2.20) coincides with the quantity ζ_{new} defined in Eq. (96) of [15].^{*4} The importance of ζ_{new} was discussed in [15]: it absorbs all terms in the action generated by a disformal transformation that explicitly depend on the time derivative of the perturbation of the lapse function. In so doing, it makes it explicitly clear that no additional degrees of freedom appear as a result of the disformal transformation. In light of the above analysis, we see that the appearance of the quantity $\zeta_{\text{new}} (= \tilde{\mathcal{R}}_c)$ is in fact very natural, as it simply corresponds to the transformed comoving curvature perturbation.

The quantity ϵ_s is a gauge-invariant quantity corresponding to the gauge-invariant perturbation of X , as seen in (2.21). It also coincides with the comoving density perturbation for a single canonical scalar field, $\epsilon_s = \delta\rho_s = \delta\rho - 3H\delta q$, where $\delta\rho = \delta X + V_\phi\delta\phi$ is the density perturbation and $\delta q = -\dot{\phi}\delta\phi$ is the velocity potential for the energy momentum tensor of the scalar field [32]. ϵ_s is also related to the intrinsic entropy perturbation of a canonical scalar field as

$$S \equiv H \left(\frac{\delta p}{\dot{p}} - \frac{\delta\rho}{\dot{\rho}} \right) = \frac{2V_\phi}{3\dot{\phi}^2(3H\dot{\phi} + 2V_\phi)} \epsilon_s. \quad (2.22)$$

So far, we have not assumed any particular scalar-tensor theory, and thus (2.20) holds for any theory. We now proceed to consider specific theories, in order to determine how ϵ_s behaves. In General Relativity, if the scalar field is the dominant energy component of the universe, then from Einstein's equations we are able to determine that ϵ_s satisfies the Poisson equation

$$-\frac{k^2}{a^2}\Psi = \frac{\epsilon_s}{2}, \quad (2.23)$$

where $\Psi \equiv \psi + a^2 H(\dot{E} - B/a)$ is the gauge-invariant Bardeen potential. As such, ϵ_s is suppressed by k^2 on large scales as long as Ψ remains finite, which implies that the difference between $\tilde{\mathcal{R}}_c$ and \mathcal{R}_c will also vanish on large scales.

It has also been shown that in a subclass of Horndeski's scalar-tensor theory ϵ_s still vanishes on superhorizon scales if the comoving curvature perturbation remains constant [27]. To the best of our knowledge, however, it has not yet been explicitly shown for the full Horndeski theory, and this is what we will now proceed to confirm. Here we simply give the result, and more details can be found in Appendix A.

Horndeski's action takes the form

$$S = \int d^4x \sqrt{-g} \mathcal{L}_H, \quad (2.24)$$

with $\mathcal{L}_H = \sum_{i=2}^5 \mathcal{L}_i$ and

$$\mathcal{L}_2 = K(\phi, X), \quad (2.25)$$

$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \quad (2.26)$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4X} [(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)], \quad (2.27)$$

$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X} [(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)]. \quad (2.28)$$

Focusing on scalar perturbations, and taking the spatial gauge $E = 0$ in (2.8), the equations of motion for A , B , ψ and $\delta\phi$ were derived in [33]. Combining the constraint equations that result from varying the second order action with respect to A and B , we are able to derive the following Poisson equation for ϵ_s

$$\epsilon_s = \frac{k^2}{a^2 H^2} \frac{\dot{\phi}^2 C_1 H^2}{C_1 A_4 - C_3 A_1} \left(A_3 \mathcal{R}_c + \frac{A_5}{H} (\Psi + \mathcal{R}_c) \right), \quad (2.29)$$

where the coefficients A_i and C_i depend only on background quantities and are given in Appendix A. In the case of General Relativity with a single canonical scalar field, where $G_3 = G_5 = 0$, $K(\phi, X) = X - V(\phi)$ and $G_4 = 1/2$, we have

$$A_1 = 6H, \quad A_3 = 2, \quad A_4 = 2X - 6H^2, \quad A_5 = -2H, \quad C_1 = 2 \quad \text{and} \quad C_3 = -2H, \quad (2.30)$$

^{*4} In fact, the two expressions do not exactly coincide, but this is due to a typo in Eq. (96) of [15], where \tilde{N} should be replaced by $N(= 1 + \delta N)$, so that to linear order the expression for ζ_{new} should be $\zeta_{\text{new}} = \zeta + \frac{\Omega_N}{\Omega} \delta N$. We thank J. Gleyzes for confirming this point.

so that (2.29) reduces to (2.23).

Given the form of (2.29), one can conclude that ϵ_s vanishes on superhorizon scales — i.e. in the limit $k \ll aH$ — so long as the coefficient of $k^2/(aH)^2$ on the right hand side is finite in this limit. Alternatively, from the momentum constraint we have

$$\epsilon_s = \frac{C_1}{C_3} \dot{\phi}^2 \dot{\mathcal{R}}_c, \quad (2.31)$$

which is given in (A11) but we repeat here for convenience. As such, we see that a sufficient condition for the vanishing of ϵ_s — and thus disformal invariance of \mathcal{R}_c — is that \mathcal{R}_c is conserved.

In the case of General Relativity plus canonical scalar field, it is well known that \mathcal{R}_c is conserved on superhorizon scales, provided that the so-called decaying mode can be neglected. Explicitly, one finds that $\dot{\mathcal{R}}_c \propto 1/(\epsilon a^3)$, where $\epsilon = \dot{\phi}^2/(2H^2)$. As such, we see that $\dot{\mathcal{R}}_c$ is indeed decaying — and therefore negligible — provided ϵ is not decaying faster than a^{-3} . This condition is satisfied in almost all standard slow-roll inflation models, where the slow-roll parameter ϵ is itself taken to be slowly varying. There is, however, a special class of inflation models — dubbed “ultra-slow-roll” models — for which extra care is needed [34–37]. In the simplest ultra-slow-roll inflation model with constant potential one finds $\epsilon \propto a^{-6}$, which means that $\dot{\mathcal{R}}_c$ is growing as a^3 . Interestingly, however, we still find that ϵ_s decays as a^{-3} , which follows from the fact that ϵ_s and $\dot{\mathcal{R}}_c$ are related by a factor of $\dot{\phi}^2 \propto a^{-6}$. As such, provided the disformal transformation is such that the factor α_{0X}/α_0 appearing in (2.20) is not growing faster than a^3 , we see that \mathcal{R}_c is disformally invariant even in the case of ultra-slow-roll inflation.

In the more general case of Horndeski’s theory we have a similar result. As was shown in [7], on superhorizon scales \mathcal{R}_c has the two independent solutions^{*5}

$$\mathcal{R}_c = \text{const.} \quad \text{and} \quad \mathcal{R}_c \propto \int^t \frac{1}{\mathcal{G}_S a^3} dt', \quad (2.32)$$

where $\mathcal{G}_S = (\Sigma/\Theta^2)\mathcal{G}_T^2 + 3\mathcal{G}_T$ and Σ , Θ and \mathcal{G}_T are as defined in Appendix A. We thus find that $\dot{\mathcal{R}}_c \propto 1/(\mathcal{G}_S a^3)$, which is decaying provided \mathcal{G}_S is not decaying faster than a^{-3} . In the case of standard slow-roll inflation we expect this to be the case, but there will be exceptions analogous to ultra-slow-roll inflation. Strictly speaking, even if the decaying mode can be neglected, in order to then conclude that $\dot{\mathcal{R}}_c = \mathcal{R}_c$ on superhorizon scales we additionally must assume that the combination $\alpha_{0X}C_1\dot{\phi}^2/(\alpha_0C_3)$ is not growing faster than $\dot{\mathcal{R}}_c$ is decaying. This again seems reasonable if we assume that background quantities are evolving slowly, but perhaps there may be some exceptions in the very general context of Horndeski’s theory. Conversely, as was the case with ultra-slow-roll inflation, even if $\dot{\mathcal{R}}_c$ is not decaying, \mathcal{R}_c may still be disformally invariant if the combination $\alpha_{0X}C_1\dot{\phi}^2/(\alpha_0C_3)$ is decaying faster than $\dot{\mathcal{R}}_c$ is growing.

To reiterate, our main conclusion is that in the context of Horndeski’s theory, a sufficient condition for the vanishing of ϵ_s — and thus disformal invariance of \mathcal{R}_c — is that \mathcal{R}_c is conserved, and this is the case on superhorizon scales so long as we can neglect the so-called decaying mode of \mathcal{R}_c . Models in which the decaying mode cannot be neglected — such as ultra-slow-roll inflation — must be considered on a case-by-case basis, but interestingly it seems that the disformal invariance of \mathcal{R}_c does not necessarily break down in such cases. For the remainder of this section we will restrict ourselves to considering models in which the decaying mode can be neglected.

Using the above results, we can argue that the comoving density perturbation ϵ_s should vanish on superhorizon scales in any theory that is disformally related to Horndeski’s theory as follows. Suppose we have two theories, theory A and theory B, that are both disformally related to an element of Horndeski’s theory, theory H. The metrics for these theories are related as

$$g_{\mu\nu}^{(A)} = \alpha^{(H \rightarrow A)} g_{\mu\nu}^{(H)} + \beta^{(H \rightarrow A)} \partial_\mu \phi \partial_\nu \phi, \quad (2.33)$$

$$g_{\mu\nu}^{(B)} = \alpha^{(H \rightarrow B)} g_{\mu\nu}^{(H)} + \beta^{(H \rightarrow B)} \partial_\mu \phi \partial_\nu \phi. \quad (2.34)$$

We can then consider a disformal transformation between theory A and theory B

$$g_{\mu\nu}^{(B)} = \alpha^{(A \rightarrow B)} g_{\mu\nu}^{(A)} + \beta^{(A \rightarrow B)} \partial_\mu \phi \partial_\nu \phi, \quad (2.35)$$

with

$$\alpha^{(A \rightarrow B)} = \frac{\alpha^{(H \rightarrow B)}}{\alpha^{(H \rightarrow A)}}, \quad (2.36)$$

$$\beta^{(A \rightarrow B)} = \beta^{(H \rightarrow B)} - \frac{\alpha^{(H \rightarrow B)}}{\alpha^{(H \rightarrow A)}} \beta^{(H \rightarrow A)}. \quad (2.37)$$

^{*5} Note that these two solutions are in fact valid on scales larger than the sound horizon of the scalar perturbation \mathcal{R}_c . Horizon crossing is defined by $c_s^2 k^2 = a^2 H^2$, where c_s is the sound speed of \mathcal{R}_c and is in general different from unity. See [7] for the general expression.

Then, the comoving curvature perturbations in these theories are related as

$$\mathcal{R}_c^{(A)} - \mathcal{R}_c^{(H)} = \frac{\alpha_{0X}^{(H \rightarrow A)}}{2\alpha_0^{(H \rightarrow A)}} \epsilon_s^{(H)}, \quad (2.38)$$

$$\mathcal{R}_c^{(B)} - \mathcal{R}_c^{(H)} = \frac{\alpha_{0X}^{(H \rightarrow B)}}{2\alpha_0^{(H \rightarrow B)}} \epsilon_s^{(H)}. \quad (2.39)$$

Here, $\alpha_{0X}^{(P \rightarrow Q)}$ is understood as a derivative with respect to $X^{(P)} \equiv g_{(P)}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. As the comoving density perturbation $\epsilon_s^{(H)}$ vanishes on superhorizon scales in Horndeski's theory, $\mathcal{R}_c^{(A)} = \mathcal{R}_c^{(B)} = \mathcal{R}_c^{(H)}$ on superhorizon scales. By considering the disformal transformation of \mathcal{R}_c between theories A and B, which gives us

$$\mathcal{R}_c^{(B)} - \mathcal{R}_c^{(A)} = \frac{\alpha_{0X}^{(A \rightarrow B)}}{2\alpha_0^{(A \rightarrow B)}} \epsilon_s^{(A)}, \quad (2.40)$$

the vanishing of the left hand side on superhorizon scales allows us to infer the vanishing of $\epsilon_s^{(A)}$ on superhorizon scales.

Indeed, we can also confirm the above statement explicitly by considering how ϵ_s transforms under a disformal transformation. Using (2.12), (2.17) and (2.21), we obtain

$$\tilde{\epsilon}_s = \frac{\alpha_0 - \alpha_{0X} \frac{\dot{\phi}^2}{2} + 2\beta_{0X} \left(\frac{\dot{\phi}^2}{2}\right)^2}{(\alpha_0 - \beta_{0X} \dot{\phi}^2)^2} \epsilon_s, \quad (2.41)$$

which recovers the result in [27] when α and β are functions of ϕ only. Interestingly, even for a general disformal transformation with X -dependent α and β , ϵ_s is disformally invariant up to a coefficient depending on background quantities. This implies that any theory disformally related to Horndeski's theory should also have vanishing ϵ_s on large scales.

Let us recall here that we are assuming the invertibility condition (2.6) is satisfied. This condition precisely guarantees that the coefficient in front of ϵ_s in (2.41) neither diverges nor vanishes. As such, the conclusion that $\tilde{\epsilon}_s$ vanishes if ϵ_s vanishes holds for any disformal transformation that is invertible. It is interesting to note that the coefficient on the right-hand side of (2.41) exactly coincides with the left-hand side of (2.5) evaluated at background level, i.e. it coincides with $\partial \tilde{X} / \partial X$. This can be expected, however, as $\tilde{\epsilon}_s$ corresponds to the gauge-invariant perturbation of \tilde{X} and ϵ_s to the gauge-invariant perturbation of X (recall (2.21)).

III. NONLINEAR ANALYSIS

In going beyond linear perturbations let us take the unitary gauge from the outset, where $\delta\phi = 0$. In this case, the spatial part of the metric is not affected by the disformal part of the disformal transformation, β , due to the fact that $\partial_i \phi = 0$. Let us take the metric with nonlinear perturbations as

$$ds^2 = -N^2 dt^2 + a^2 e^{2\mathcal{R}_c} \gamma_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \quad (3.1)$$

where

$$\gamma_{ij} = e^{2\partial_i \partial_j E + \partial_i F_j + \partial_j F_i + h_{ij}} \quad (3.2)$$

and F_i and h_{ij} once again satisfy $F_{i,i} = h_{ii} = h_{ij,i} = 0$. Note also that N^i contains both scalar and vector components, and that spatial indices should be raised and lowered with γ^{ij} and γ_{ij} , respectively. If we further parameterise $\alpha(\phi, X)$ as $\alpha \equiv \alpha_0 e^{2\Delta\alpha}$, where α_0 corresponds to the background part of α and $e^{2\Delta\alpha}$ contains nonlinear perturbations from this background value, then we see that in the gauge $\delta\phi = 0$ the metric transforms as

$$\begin{aligned} \tilde{g}_{00} &= -\alpha N^2 + \beta \dot{\phi}^2 + \alpha_0 a^2 e^{2\mathcal{R}_c + 2\Delta\alpha} \gamma_{ij} N^i N^j, \\ \tilde{g}_{0i} &= \alpha_0 a^2 e^{2\mathcal{R}_c + 2\Delta\alpha} \gamma_{ij} N^j, \\ \tilde{g}_{ij} &= \alpha_0 a^2 e^{2\mathcal{R}_c + 2\Delta\alpha} \gamma_{ij}, \end{aligned} \quad (3.3)$$

from which we deduce

$$\tilde{N}^2 = \alpha N^2 - \beta \dot{\phi}^2, \quad \tilde{a}^2 = \alpha_0 a^2, \quad \tilde{\mathcal{R}}_c = \mathcal{R}_c + \Delta\alpha, \quad (3.4)$$

whilst E , F_i , N^i and h_{ij} remain unchanged at nonlinear level, which is consistent with the result at linear level in the previous section. As such, the vector and tensor perturbations are invariant at the nonlinear level.

In analysing the last relation in (3.4), note that as β is irrelevant for the transformation law of \mathcal{R}_c , the situation is equivalent to determining how \mathcal{R}_c transforms under a conformal transformation. As such, in the case that $\alpha = \alpha(\phi)$ our conclusion is the same as that reached in [26]: in the unitary gauge $\delta\phi = 0$ so that $\Delta\alpha = 0$, meaning that \mathcal{R}_c is invariant. This is also in agreement with [28] and – at the linear level – with the results of §II and [27].

In contrast, when we allow for an X -dependence of α , even in the unitary gauge we find that $\Delta\alpha \neq 0$ as a result of the dependence of X on N . Explicitly, in the unitary gauge we have $\alpha = \alpha(\phi, \dot{\phi}^2/(2N^2))$. As such, we see that α will only coincide with its background value, hence giving $\Delta\alpha = 0$ and $\tilde{\mathcal{R}}_c = \mathcal{R}_c$, when the perturbation of the lapse function vanishes. At linear order, this condition corresponds to $A = 0$, which is thus consistent with the requirement found in §II that ϵ_s must vanish if we are to have $\tilde{\mathcal{R}}_c = \mathcal{R}_c$, as in the unitary gauge we have $\epsilon_s = -\dot{\phi}^2 A$.

In order to aid an intuitive understanding of this condition, let us define the proper time τ as $d\tau = Ndt$. Starting with the definition of ϵ_s at linear order, we can see that it can be rewritten as

$$\epsilon_s = \partial_\tau \phi \partial_\tau^2 \phi \left(\frac{\delta(\partial_\tau \phi)}{\partial_\tau^2 \phi} - \frac{\delta\phi}{\partial_\tau \phi} \right), \quad (3.5)$$

where the term in brackets corresponds to the relative entropy perturbation between $\partial_\tau \phi$ and ϕ . We can thus see that ϵ_s vanishes – in turn giving $\tilde{\mathcal{R}}_c = \mathcal{R}_c$ – when $\partial_\tau \phi = f(\phi)$, where $f(\phi)$ is some function of ϕ . Similarly, turning to the nonlinear case, we see that in terms of τ we can write $\alpha = \alpha(\phi, (\partial_\tau \phi)^2/2)$. Imposing $\partial_\tau \phi = f(\phi)$ means that in the unitary gauge α is equal to its background value, which in turn gives us $\Delta\alpha = 0$ and $\tilde{\mathcal{R}}_c = \mathcal{R}_c$. The condition $\partial_\tau \phi = f(\phi)$ is familiar to us as the condition for an attractor regime (see e.g. [38]), and we thus conclude that in the unitary gauge and an attractor regime the curvature perturbation is disformally invariant at the nonlinear level. Note that the requirement to be in an attractor regime is not as restrictive as it may sound. Indeed, the vast majority of standard inflationary models satisfy this condition.

It has been shown in [39] that even at the nonlinear level the comoving curvature perturbation has a mode that remains constant on superhorizon scales in Horndeski's theory. Provided that the other so-called decaying mode can be neglected, this allows us to conclude that on superhorizon scales the nonlinear comoving curvature perturbation is both conserved and disformally invariant in any theory that is disformally related to Horndeski's if one is in the attractor regime.

IV. THE NUMBER OF DEGREES OF FREEDOM IN THEORIES DISFORMALLY RELATED TO HORNDESKI'S

In the preceding sections we have investigated the transformation properties of the comoving curvature perturbation both at the linear and nonlinear level, with our conclusions holding for any scalar-tensor theory that is disformally related to Horndeski's theory. In this section we discuss whether or not such theories are well behaved in the sense that they do not suffer from so-called Ostrogradsky instabilities. Such instabilities generically arise in theories possessing equations of motion that are higher than second order in time derivatives, indicating the presence of additional ghost-like degrees of freedom. As we will see below, the equations of motion derived from theories disformally related to Horndeski's theory do seemingly contain higher-order derivatives, thus suggesting that they are unhealthy. However, this is somewhat at odds with our expectation given that the theories are simply related to instability-free Horndeski's theory by a redefinition of fields. In the following we try to address this apparent contradiction.

The action associated with the aforementioned class of theories can be written in the following form

$$S = \int d^4x \sqrt{-\tilde{g}} \mathcal{L}_H(\tilde{g}_{\mu\nu}, \phi) + \int d^4x \sqrt{-g} \mathcal{L}_m(g_{\mu\nu}), \quad (4.1)$$

where we have Horndeski's Lagrangian, \mathcal{L}_H , written in terms of the metric $\tilde{g}_{\mu\nu}$ and matter is minimally coupled to the metric $g_{\mu\nu}$. The metric $\tilde{g}_{\mu\nu}$ is disformally related to $g_{\mu\nu}$ as in (1.1). The frame defined by $\tilde{g}_{\mu\nu}$ is the ‘‘Horndeski frame’’, in which the gravitational Lagrangian coincides with that of Horndeski's theory and matter is non-minimally coupled to the scalar field through $g_{\mu\nu} = (\tilde{g}_{\mu\nu} - \beta \partial_\mu \phi \partial_\nu \phi)/\alpha$. However, we are interested in the equations of motion in the Jordan frame, defined by $g_{\mu\nu}$, in which matter is not coupled to the scalar field. Varying the above action with

respect to $g_{\mu\nu}$ and ϕ yields their equations of motion,

$$\alpha \mathcal{E}_H^{\mu\nu} + \frac{1}{2} \mathcal{E}_H^{\rho\sigma} (\alpha_X g_{\rho\sigma} + \beta_X \partial_\rho \phi \partial_\sigma \phi) \partial^\mu \phi \partial^\nu \phi + \frac{1}{2\alpha\sqrt{\alpha(\alpha-2X\beta)}} T_m^{\mu\nu} = 0, \quad (4.2)$$

$$\begin{aligned} \nabla_\mu \left[\alpha \sqrt{\alpha(\alpha-2X\beta)} \{ \mathcal{E}_H^{\rho\sigma} (\alpha_X g_{\rho\sigma} + \beta_X \partial_\rho \phi \partial_\sigma \phi) \partial^\mu \phi - 2\beta \mathcal{E}_H^{\mu\nu} \partial_\nu \phi \} \right] \\ + \alpha \sqrt{\alpha(\alpha-2X\beta)} \left[\mathcal{E}_H^{\rho\sigma} (\alpha_\phi g_{\rho\sigma} + \beta_\phi \partial_\rho \phi \partial_\sigma \phi) + \mathcal{E}_H^{(\phi)} \right] = 0, \end{aligned} \quad (4.3)$$

where $\mathcal{E}_H^{\mu\nu}$ and $\mathcal{E}^{(\phi)}$ are determined by varying Horndeski's Lagrangian with respect to $\tilde{g}_{\mu\nu}$ and ϕ [7, 39], namely, $\delta(\sqrt{-\tilde{g}}\mathcal{L}_H) = \sqrt{-\tilde{g}}(\mathcal{E}_H^{\mu\nu} \delta\tilde{g}_{\mu\nu} + \mathcal{E}^{(\phi)} \delta\phi)$, and $T_m^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}}$ denotes the energy-momentum tensor associated with the matter Lagrangian. Note that here $\partial^\mu \phi = g^{\mu\alpha} \partial_\alpha \phi$. As $\mathcal{E}_H^{\mu\nu}$ and $\mathcal{E}^{(\phi)}$ are the equations of motion for Horndeski's theory, they contain at most second order derivatives of ϕ and $\tilde{g}_{\mu\nu}$, meaning that they also contain at most second order derivatives of $g_{\mu\nu}$. However, as $\tilde{g}_{\mu\nu}$ contains $X = -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / 2$, in principle it is possible for $\mathcal{E}_H^{\mu\nu}$ and $\mathcal{E}^{(\phi)}$ to contain up to third order derivatives of ϕ . As such, we see that (4.2) will contain up to second order derivatives of $g_{\mu\nu}$ but potentially third order derivatives of ϕ . Similarly, (4.3) will contain up to third order derivatives of $g_{\mu\nu}$ and fourth order derivatives of ϕ .

As already mentioned, it is reasonable to expect that the appearance of seemingly dangerous higher order derivatives in (4.2) and (4.3) may be spurious, as we know that our theory is related to a healthy theory by a field redefinition. Indeed, following the analysis of [31, 40], it is possible to show that the gravitational equations of motion (4.2) are equivalent to the equations of motion obtained by varying the action with respect to $\tilde{g}_{\mu\nu}$ so long as the transformation (1.1) is invertible. Explicitly, we obtain (see Appendix B for details)

$$\mathcal{E}_H^{\mu\nu} + \frac{1}{2} T_H^{\mu\nu} = 0, \quad (4.4)$$

where $T_H^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}}$ and is given explicitly in terms of $T_m^{\mu\nu}$ as

$$T_H^{\mu\nu} = \frac{1}{\alpha^2 \sqrt{\alpha(\alpha-2X\beta)}} \left[T_m^{\mu\nu} - \frac{\partial^\mu \phi \partial^\nu \phi}{2(\alpha + 2X^2 \beta_X - X \alpha_X)} T_m^{\rho\sigma} (\alpha_X g_{\rho\sigma} + \beta_X \partial_\rho \phi \partial_\sigma \phi) \right]. \quad (4.5)$$

Note that $T_H^{\mu\nu}$ contains at most first order derivatives of ϕ . On substituting the above expression for $\mathcal{E}_H^{\mu\nu}$ into (4.3), we obtain

$$\begin{aligned} \alpha \sqrt{\alpha(\alpha-2X\beta)} \mathcal{E}_H^{(\phi)} + \nabla_\mu \left[\frac{\beta}{\alpha} T_m^{\mu\nu} \partial_\nu \phi - \frac{\alpha-2X\beta}{2\alpha(\alpha+2X^2\beta_X-X\alpha_X)} (\alpha_X g_{\rho\sigma} + \beta_X \partial_\rho \phi \partial_\sigma \phi) T_m^{\rho\sigma} \partial^\mu \phi \right] \\ + \frac{T_m^{\rho\sigma}}{2} \left[\frac{2X^2 \alpha_X \beta_\phi - \alpha_\phi (\alpha + 2X^2 \beta_X)}{\alpha(\alpha + 2X^2 \beta_X - X \alpha_X)} g_{\rho\sigma} + \frac{\partial_\rho \phi \partial_\sigma \phi}{\alpha + 2X^2 \beta_X} \left(-\beta_\phi + \frac{X \beta_X [2X^2 \alpha_X \beta_\phi - \alpha_\phi (\alpha + 2X^2 \beta_X)]}{\alpha(\alpha + 2X^2 \beta_X - X \alpha_X)} \right) \right] = 0. \end{aligned} \quad (4.6)$$

If $T_m^{\mu\nu}$ does not contain any second order derivatives of $g_{\mu\nu}$, then we see that (4.4) and (4.6) both contain at most second order derivatives of $g_{\mu\nu}$. Moreover, they both contain at most third order derivatives of ϕ , which result from the second order derivatives of $\tilde{g}_{\mu\nu}$ appearing in $\mathcal{E}_H^{\mu\nu}$ and $\mathcal{E}_H^{(\phi)}$. Note that, as a result of the kinetic mixing between the scalar field and the metric in (4.3), the equation of motion (4.6) contains mixing terms between ϕ and the matter energy-momentum tensor and its derivatives. The rich structure of kinetic mixing found in general scalar-tensor theories has many interesting consequences, as discussed in [41].

Although the situation has been marginally improved in moving from (4.2) and (4.3) to (4.4) and (4.6), we still have third order derivatives of ϕ appearing, which would normally indicate the presence of dangerous additional degrees of freedom. One can superficially remove these third order derivatives by using the trace of the gravitational equations (4.4) – sometimes referred to as a “hidden constraint” – to find an expression for ϕ in terms of lower order derivatives of ϕ and $g_{\mu\nu}$, which can then be substituted into (4.4) [8, 12]. However, as recently highlighted in [16], this does not offer a rigorous proof that the number of degrees of freedom is unaffected by the disformal transformation, as the trace equation itself is a dynamical equation that must still be satisfied.

A more rigorous approach, which is in the same vein as that mentioned above and has been demonstrated in [16], is to find a suitable linear combination of the gravitational equations of motion (4.4) that does not contain ϕ and allows you to express ϕ in terms of at most first order time derivatives of ϕ and $g_{\mu\nu}$. On taking the derivative of this relation we are then able to find an expression for $\dot{\phi}$ in terms of at most second order time derivatives of ϕ and $g_{\mu\nu}$, which can be substituted back into the original equations. The difference here is that the special combination of gravitational equations used does not itself involve third order derivatives of ϕ . In [16], Deffayet *et al.* showed

that such a combination does indeed exist for a very general class of models in the context of covariantized Galileons. More specifically, they considered theories that in four spacetime dimensions are equivalent to (2.24) but with the replacements $G_{4X}(\phi, X) \rightarrow F_4(\phi, X)$ and $G_{5X}(\phi, X) \rightarrow F_5(\phi, X)$, where F_4 and F_5 are arbitrary functions of ϕ and X . When evaluated in the unitary gauge, this Lagrangian is equivalent to the Lagrangian of the theories beyond Horndeski's considered in [9], and as mentioned in the introduction, the structure of this Lagrangian is preserved under disformal transformations with $\alpha = \alpha(\phi)$ and $\beta = \beta(\phi, X)$. The models under consideration here, however, are related to Horndeski's theory by a more general disformal transformation with $\alpha = \alpha(\phi, X)$ and $\beta = \beta(\phi, X)$, meaning that they do not belong to the class analysed by Deffayet *et al.* As such, the appropriate combination of gravitational equations with which we can express $\dot{\phi}$ in terms of first order time derivatives of ϕ and $g_{\mu\nu}$ remains to be found.

Another rigorous approach to determine the number of degrees of freedom of the theory is to perform a Hamiltonian analysis. Such an analysis has been performed for the newly-discovered theories beyond Horndeski's in Refs. [11, 12, 14], but in these analyses the unitary gauge was taken from the outset, which could affect the outcome of the degrees-of-freedom counting [16]. A Hamiltonian analysis that does not rely on fixing the gauge has been performed for an example Lagrangian in [16], but its generalisation is yet to be carried out.

In the context of theories that are disformally related to Horndeski's, we are particularly interested in how the Hamiltonian analysis of a theory is affected by a disformal transformation. In the case of a conformal transformation, where $\beta = 0$ and $\alpha = \alpha(\phi)$, it has been demonstrated in the context of $f(R)$ gravity that the transformation simply gives rise to a canonical transformation of variables, thus rendering the Hamiltonian analysis unchanged [42]. More general metric transformations – which include disformal transformations of the form (1.1) – were also considered recently in [29], where they reached a similar conclusion; we will comment further on their results shortly. Whilst a complete analysis of how the Hamiltonian analysis of a theory is affected by disformal transformations is beyond the scope of this paper, it is nevertheless possible to see why we might expect the degrees-of-freedom counting to remain unchanged.

The key distinguishing feature of general disformal transformations taking the form (1.1) as opposed to conformal transformations with $\beta = 0$ and $\alpha = \alpha(\phi)$ is that they induce higher order derivatives of some of the fields in the theory at the level of the action.^{*6} However, the fact that these higher order derivatives always appear in a certain combination – as we will see with the help of a toy model below – means that they are always associated with additional primary constraints in the Hamiltonian analysis. If these constraints are first class, or second class and give rise to secondary second class constraints, then they will remove the spurious degrees of freedom associated with the higher derivatives appearing in the action as a result of the transformation. Indeed, this is the case in the example model considered in Sec. III of [29], where a derivative of the lapse, N , induced by the disformal transformation is always associated with a derivative of the spatial metric, γ_{ij} , leading to a primary constraint involving the corresponding canonical momenta π_N and π^{ij} . The “exorcising of Ostrogradsky's ghost” with constraints was also discussed in [43].

In order to demonstrate the above idea, let us consider a simple toy model consisting of two degrees of freedom $x(t)$ and $z(t)$, with $\mathcal{L}_0 = \mathcal{L}_0(x, \dot{x}, z, \dot{z})$. Here we have in mind that x represents the metric degrees of freedom $\tilde{g}_{\mu\nu}$ while z the scalar field ϕ . Assuming this Lagrangian to be regular means that we have two degrees of freedom and require four initial conditions. Analogous to a disformal transformation, we then consider that $x = x(y, z, \dot{z})$, where $y(t)$ represents the disformally related metric $g_{\mu\nu}$, and the transformation depends on the analogues of ϕ and X , z and \dot{z} respectively. In terms of this new set of variables we have $\mathcal{L}_0 = \mathcal{L}_0(y, \dot{y}, z, \dot{z}, \ddot{z})$, i.e. due to the appearance of \ddot{z} in the transformation we have picked up a dependence of \mathcal{L}_0 on \ddot{z} . Following the constrained Ostrogradsky approach to systems with higher order derivatives – see e.g. [44, 45] – we introduce an auxiliary degree of freedom as $w = \dot{z}$, which we impose by adding a Lagrange multiplier to the Lagrangian, i.e. we have

$$\mathcal{L} = \mathcal{L}_0(y, \dot{y}, z, w, \dot{w}) + \lambda(w - \dot{z}). \quad (4.7)$$

This Lagrangian depends on four degrees of freedom and contains up to first order derivatives, so in general eight initial conditions are required. On calculating the canonical momenta we obtain

$$p_y = \frac{\partial \mathcal{L}_0}{\partial \dot{y}}, \quad p_z = -\lambda, \quad p_w = \frac{\partial \mathcal{L}_0}{\partial \dot{w}} \quad \text{and} \quad p_\lambda = 0, \quad (4.8)$$

meaning that we have two obvious primary constraints, namely

$$\Phi_1 = p_z + \lambda \approx 0 \quad \text{and} \quad \Phi_2 = p_\lambda \approx 0. \quad (4.9)$$

^{*6} To be more specific, we are interested in the case where both α and β have an X -dependence. As discussed previously, in the case where $\alpha = \alpha(\phi)$ and $\beta = \beta(\phi)$ Horndeski's theory is mapped onto itself, and we already know that Horndeski's theory propagates only three degrees of freedom. Similarly, in the case where $\alpha = \alpha(\phi)$ and $\beta = \beta(\phi, X)$, Horndeski's theory will map onto a subclass of the theories introduced in [9], which were shown to be healthy without making use of the unitary gauge in [16].

Note that these two constraints result from the introduction of the auxiliary degree of freedom, and in this sense do not correspond to the “additional primary constraints” referred to above. In order to see how these additional primary constraints appear we note that

$$p_y = \frac{\partial \mathcal{L}_0}{\partial \dot{y}} = \frac{\partial \mathcal{L}_0}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \dot{y}} = \frac{\partial \mathcal{L}_0}{\partial \dot{x}} \frac{\partial x}{\partial y} \quad \text{and} \quad p_w = \frac{\partial \mathcal{L}_0}{\partial \dot{w}} = \frac{\partial \mathcal{L}_0}{\partial \dot{z}} = \frac{\partial \mathcal{L}_0}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \dot{z}} = \frac{\partial \mathcal{L}_0}{\partial \dot{x}} \frac{\partial x}{\partial \dot{z}}, \quad (4.10)$$

so that we find the additional primary constraint

$$\Phi_3 = p_y - F(y, z, w)p_w \approx 0, \quad \text{where} \quad F(y, z, w) \equiv \frac{\partial x}{\partial y} \frac{1}{\partial x / \partial \dot{z}}. \quad (4.11)$$

On constructing the total Hamiltonian and requiring that the constraints Φ_i be preserved under time evolution we find an additional secondary constraint, which we label Φ_4 . Requiring Φ_4 to be preserved under time evolution does not give rise to any additional secondary constraints, and we find that all four constraints are second class. As such, although we started with four degrees of freedom requiring eight initial conditions, we found four second class constraints, meaning that we only have to specify four initial conditions, corresponding to only two degrees of freedom. This is in agreement with the fact that we had two degrees of freedom present in the original Lagrangian $\mathcal{L}_0(x, \dot{x}, z, \dot{z})$.

The key point in the above analysis was the appearance of the third primary constraint Φ_3 . As alluded to earlier, this arose due to the fact that \dot{z} only appears in the Lagrangian in a specific combination with \dot{y} , which is a consequence of the form of the analogue of the disformal transformation $x = x(y, z, \dot{z})$.

In fact, the toy model considered above falls into the more general class of models and transformations considered in [29]. To see this we introduce two auxiliary degrees of freedom, which allows us to re-write the Lagrangian in a form that is linear in velocities, i.e. we write

$$\mathcal{L} = \mathcal{L}_0(x, u, z, w) + \lambda_1(u - \dot{x}) + \lambda_2(w - \dot{z}). \quad (4.12)$$

When written in terms of w , the transformation no longer depends on the velocity of any of the fields, i.e. we have $x = x(y, z, w)$. This form of transformation and the form of action given in (4.12) are then of the form considered by Domènech *et al.* in [29]. Assuming that the transformation $x = x(y, z, w)$ is invertible, they show that such a transformation is a canonical transformation, so that the set of constraints and constraint algebra are left unchanged, which in turn means that the number of degrees of freedom remains the same. Note that the analysis of Domènech *et al.* does not rely on the original Lagrangian being regular, and so offers a general proof that the number of degrees of freedom is unaffected on making a disformal transformation, so long as the transformation is invertible. We thus conclude that theories disformally related to Horndeski’s theory via an invertible disformal transformation of the form (1.1) are healthy, in the sense that they propagate only three degrees of freedom.

V. CONCLUSION

In this paper we have examined how the comoving curvature perturbation transforms under the general disformal transformations of the metric given in (1.1). We began by considering linear perturbations, and showed that whilst the vector and tensor perturbation are invariant, the comoving curvature perturbation is not identically invariant under a general disformal transformation. The difference between disformally related curvature perturbations is given in (2.20), and is written in terms of the gauge-invariant comoving density perturbation ϵ_s associated with a single canonical scalar field. In the context of Horndeski’s theory we used the Hamiltonian and momentum constraints to derive a Poisson equation for ϵ_s , and found that a sufficient condition for the vanishing of ϵ_s is that \mathcal{R}_c is conserved, which is the case on superhorizon scales so long as the so-called decaying mode of \mathcal{R}_c can be neglected. As such, we concluded that the comoving curvature perturbation is disformally invariant on superhorizon scales for any theory that is disformally related to Horndeski’s theory, provided that the decaying mode of \mathcal{R}_c can be neglected. Using this result, we saw that under the same mild assumption the comoving density perturbation is suppressed on superhorizon scales for any theory that is disformally related to Horndeski’s theory, which we were also able to derive explicitly from the transformation rule for ϵ_s under disformal transformations. The relation between $\tilde{\epsilon}_s$ and ϵ_s , given in (2.41), shows that $\tilde{\epsilon}_s$ is equal to ϵ_s up to a proportionality coefficient that depends on background quantities. The coefficient is finite so long as the condition given in (2.6) for the invertibility of the disformal transformation is satisfied. Based on these findings, we conclude that in most cases of interest in the context of inflation, we are free to work in any disformally related frame when we wish to calculate the superhorizon curvature and tensor perturbations that are required in making predictions for inflationary observables. This is an extension of the conformal invariance of the curvature and tensor perturbations that is frequently exploited in the context of simple scalar-tensor theories, where it

is often much easier to perform calculations in one frame than in another. One can therefore expect that the disformal invariance of perturbations will also prove to be very useful in this wider class of theories.

Using the unitary gauge, we also considered the comoving curvature perturbation at full nonlinear order, and found that it is invariant under disformal transformations if we assume that an attractor regime has been reached, where $\partial_\tau \phi = f(\phi)$. In the context of Horndeski's theory, it is known that on superhorizon scales the nonlinear curvature perturbation is conserved in this regime so long as the so-called decaying mode can be neglected, which implies that on superhorizon scales the nonlinear comoving curvature perturbation is both disformally invariant and conserved in any theory that is disformally related to Horndeski's, provided $\partial_\tau \phi = f(\phi)$ and the decaying mode can be neglected. In addition, we found that the vector and tensor perturbation are invariant under a general disformal transformation at the nonlinear level.

Finally, we discussed the healthiness of theories disformally related to Horndeski's in relation to the presence of Ostrogradsky instabilities. Focusing on a toy model, we saw that the appearance of higher derivatives in an action that results from making a field transformation involving time derivatives of some of the fields is accompanied by the appearance of second class primary and secondary constraints in the Hamiltonian analysis. These constraints therefore remove the spurious additional degrees of freedom so that, as one would naively expect, the number of degrees of freedom is unaffected by the field redefinition. This is in agreement with the results of a recent analysis by Domènech *et al.* in [29], where it was shown that the number of degrees of freedom is indeed unaffected by disformal transformations, so long as the transformations are invertible.

Note added: Our results are in agreement with those presented in Ref. [46], which appeared on arXiv at the same time as the present article and contains some material that overlaps with the nonlinear analysis we presented in §III.

ACKNOWLEDGMENTS

We would like to thank A. Naruko, M. Sasaki and Y. Watanabe for useful discussions related to their work [46] prior to submission. We would also like to thank G. Domènech, J. Gleyzes, M. Karčiauskas, M. Minamitsuji, T. Suyama, D. Yamauchi and M. Zumalacárregui for helpful discussions and comments. Finally, we would like to thank the anonymous referee, whose comments and questions have helped us to improve the manuscript. This work was partially supported by Japan Society for the Promotion of Science (JSPS) Postdoctoral Fellowships for Research Abroad (H.M.) and JSPS Grant-in-Aid for Scientific Research (B) No. 23340058 (J.W.).

Appendix A: Vanishing of ϵ_s in Horndeski's theory

In this appendix we derive the Poisson equation (2.29) in Horndeski's theory, which is used in §II to argue that under reasonable assumptions the comoving density perturbation ϵ_s , defined in (2.21), is suppressed on superhorizon scales. This result is a generalization of the result obtained in [27] for a subclass of Horndeski's theory with $G_4 = 1/2$, $G_5 = 0$.

Taking the action (2.24), assuming a flat FLRW background with line element of the form $ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j$ and varying the action with respect to $N(t)$ gives us one of the background equations of motion [33]

$$\mathcal{E} \equiv \sum_i \mathcal{E}_i = 0, \quad (\text{A1})$$

where

$$\mathcal{E}_2 = 2XK_X - K, \quad (\text{A2})$$

$$\mathcal{E}_3 = 6X\dot{\phi}HG_{3X} - 2XG_{3\phi}, \quad (\text{A3})$$

$$\mathcal{E}_4 = -6H^2G_4 + 24H^2X(G_{4X} + XG_{4XX}) - 12HX\dot{\phi}G_{4\phi X} - 6H\dot{\phi}G_{4\phi}, \quad (\text{A4})$$

$$\mathcal{E}_5 = 2H^3X\dot{\phi}(5G_{5X} + 2XG_{5XX}) - 6H^2X(3G_{5\phi} + 2XG_{5\phi X}). \quad (\text{A5})$$

Whilst there are two additional background equations corresponding to a variation with respect to $a(t)$ and $\phi(t)$, we will not use them in the following analysis, so we omit them here.

Turning next to linear perturbations, we take a line element of the form given in (2.8), but focus on the scalar perturbations. In addition, we fix the spatial gauge such that $E = 0$. The equations of motion for the perturbations are obtained by expanding the action (2.24) to second order and varying it with respect to each of the perturbation

variables. Here we will only need two of the equations – the two constraint equations obtained by varying the action with respect to A and B – and they take the form [33]

$$-A_1\dot{\psi} + A_4A + \frac{k^2}{a^2}(-A_3\psi - aBA_5) = -A_2\dot{\phi} - A_6\frac{k^2}{a^2}\delta\phi + \mu\delta\phi, \quad (\text{A6})$$

$$-C_1\dot{\psi} + C_3A = -C_2\dot{\phi} - C_4\delta\phi, \quad (\text{A7})$$

where the coefficients are dependent on background quantities and are given as

$$\begin{aligned} A_1 &= 6\Theta, & A_2 &= -\frac{2(\Sigma + 3H\Theta)}{\dot{\phi}}, & A_3 &= 2\mathcal{G}_T, & A_4 &= 2\Sigma, \\ A_5 &= -2\Theta, & A_6 &= \frac{2(\Theta - H\mathcal{G}_T)}{\dot{\phi}}, & \mu &= \mathcal{E}_\phi, \\ C_1 &= 2\mathcal{G}_T, & C_2 &= \frac{2(\Theta - H\mathcal{G}_T)}{\dot{\phi}}, & C_3 &= -2\Theta, & C_4 &= \frac{1}{\dot{\phi}^2} \left[2(H\ddot{\phi} - \dot{H}\dot{\phi})\mathcal{G}_T - 2\ddot{\phi}\Theta \right], \end{aligned} \quad (\text{A8})$$

with Σ , Θ and \mathcal{G}_T being defined as

$$\Sigma = X \frac{\partial \mathcal{E}}{\partial X} + \frac{1}{2} H \frac{\partial \mathcal{E}}{\partial H}, \quad \Theta = -\frac{1}{6} \frac{\partial \mathcal{E}}{\partial H}, \quad \mathcal{G}_T = 2 \left[G_4 - 2XG_{4X} - X \left(H\dot{\phi}G_{5X} - G_{5\phi} \right) \right]. \quad (\text{A9})$$

The easiest way to obtain (2.29) is to consider the unitary gauge, where $\delta\phi = 0$. In this gauge we have $\psi = -\mathcal{R}_c$, $A = -\epsilon_s/\dot{\phi}^2$ and $B = -(\Psi + \mathcal{R}_c)/(aH)$ and Eqs. (A6) and (A7) thus reduce to

$$A_1\dot{\mathcal{R}}_c - A_4\frac{\epsilon_s}{\dot{\phi}^2} + \frac{k^2}{a^2} \left(A_3\mathcal{R}_c + \frac{A_5}{H}(\Psi + \mathcal{R}_c) \right) = 0, \quad (\text{A10})$$

$$C_1\dot{\mathcal{R}}_c - C_3\frac{\epsilon_s}{\dot{\phi}^2} = 0. \quad (\text{A11})$$

Eliminating $\dot{\mathcal{R}}_c$ from these equations we arrive at (2.29).

One can, of course, derive (2.29) without having to fix the gauge. To see this explicitly, we eliminate $\dot{\psi}$ from Eqs. (A6) and (A7) to obtain

$$\left(A_2 - \frac{A_1C_2}{C_1} \right) \dot{\phi} + \left(A_4 - \frac{A_1C_3}{C_1} \right) A - \left(\frac{A_1C_4}{C_1} + \mu \right) \delta\phi = \frac{k^2}{a^2} (A_3\psi + aBA_5 - A_6\delta\phi). \quad (\text{A12})$$

Considering the left-hand side (l.h.s.) of (A12) first, it can be re-written as

$$\text{l.h.s.} = \frac{1}{\dot{\phi}^2} \left(\frac{A_1C_3}{C_1} - A_4 \right) \left[\dot{\phi}^2 \left(\frac{A_2C_1 - A_1C_2}{A_1C_3 - A_4C_1} \right) \dot{\phi} - \dot{\phi}^2 A - \dot{\phi}^2 \left(\frac{A_1C_4 + \mu C_1}{A_1C_3 - A_4C_1} \right) \delta\phi \right]. \quad (\text{A13})$$

Using (A8) one can then confirm that

$$\frac{A_2C_1 - A_1C_2}{A_1C_3 - A_4C_1} = \frac{1}{\dot{\phi}}. \quad (\text{A14})$$

In evaluating the coefficient of the $\delta\phi$ term we note that as a result of the background equation of motion $\mathcal{E} = 0$ we have

$$\frac{d\mathcal{E}}{dt} = \frac{\partial \mathcal{E}}{\partial \phi} \dot{\phi} + \frac{\partial \mathcal{E}}{\partial X} \dot{X} + \frac{\partial \mathcal{E}}{\partial H} \dot{H} = 0, \quad (\text{A15})$$

from which we are able to obtain an expression for $\mu = \mathcal{E}_\phi$. Using this result, and noting that $\dot{X} = 2X\ddot{\phi}/\dot{\phi}$, we find

$$\frac{A_1C_4 + \mu C_1}{A_1C_3 - A_4C_1} = \frac{\ddot{\phi}}{\dot{\phi}^2}. \quad (\text{A16})$$

Altogether we thus obtain

$$\text{l.h.s.} = \frac{1}{\dot{\phi}^2} \left(\frac{A_1C_3}{C_1} - A_4 \right) \epsilon_s. \quad (\text{A17})$$

Turning next to the right-hand side (r.h.s.) of (A12), we are able to re-write it as

$$\text{r.h.s.} = -\frac{k^2}{a^2} \left[A_3 \mathcal{R}_c + \frac{A_5}{H} (\Psi + \mathcal{R}_c) + \left(\frac{A_3 H}{\dot{\phi}} + A_6 + \frac{A_5}{\dot{\phi}} \right) \delta\phi \right]. \quad (\text{A18})$$

Using (A8) one can then show that $A_3 H + \dot{\phi} A_6 + A_5 = 0$. As such, we have recovered (2.29) without fixing the time slicing.

Appendix B: Equivalence of equations of motion in disformally related frames

In this appendix, following the analyses of [31, 40], we show that the gravitational equations given in (4.2) are equivalent to the gravitational equations of motion determined by varying (4.1) with respect to $\tilde{g}_{\mu\nu}$, so long as the disformal transformation relating $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ is invertible, namely the invertibility condition (2.6) is satisfied.

We will use the following relation between $T_m^{\mu\nu}$ and $T_H^{\mu\nu}$, i.e. the inverse relation of (4.5), which is given as

$$T_m^{\mu\nu} = \alpha \sqrt{\alpha(\alpha - 2X\beta)} \left\{ \alpha T_H^{\mu\nu} + \frac{\partial^\mu \phi \partial^\nu \phi}{2} (\alpha_X g_{\alpha\beta} + \beta_X \partial_\alpha \phi \partial_\beta \phi) T_H^{\alpha\beta} \right\}. \quad (\text{B1})$$

Here, the coefficient in front of the large bracket on the right hand side does not vanish as (2.6) is satisfied. Using the above relation it is possible to rewrite (4.2) as

$$\alpha \left(\mathcal{E}_H^{\mu\nu} + \frac{1}{2} T_H^{\mu\nu} \right) + \frac{1}{2} \left(\mathcal{E}_H^{\rho\sigma} + \frac{1}{2} T_H^{\rho\sigma} \right) (\alpha_X g_{\rho\sigma} + \beta_X \partial_\rho \phi \partial_\sigma \phi) \partial^\mu \phi \partial^\nu \phi = 0. \quad (\text{B2})$$

Contracting this equation once with $g_{\mu\nu}$ and once with $\partial_\mu \phi \partial_\nu \phi$ we obtain the two equations

$$(\alpha - X\alpha_X)P - X\beta_X Q = 0, \quad (\text{B3})$$

$$2X^2\alpha_X P + (\alpha + 2X^2\beta_X)Q = 0, \quad (\text{B4})$$

where

$$P = \left(\mathcal{E}_H^{\mu\nu} + \frac{1}{2} T_H^{\mu\nu} \right) g_{\mu\nu} \quad \text{and} \quad Q = \left(\mathcal{E}_H^{\mu\nu} + \frac{1}{2} T_H^{\mu\nu} \right) \partial_\mu \phi \partial_\nu \phi. \quad (\text{B5})$$

The solution to these two equations is $P = 0$ and $Q = 0$, so long as the matrix M , given as

$$M = \begin{pmatrix} \alpha - X\alpha_X & -X\beta_X \\ 2X^2\alpha_X & \alpha + 2X^2\beta_X \end{pmatrix} \quad (\text{B6})$$

is invertible. Substituting $P = Q = 0$ into (B2), and assuming $\alpha \neq 0$, we thus recover

$$\mathcal{E}_H^{\mu\nu} + \frac{1}{2} T_H^{\mu\nu} = 0, \quad (\text{B7})$$

i.e. we have recovered the gravitational equations obtained on minimising (4.1) with respect to $\tilde{g}_{\mu\nu}$. The case when M is not invertible corresponds to

$$\alpha(\alpha - X\alpha_X + 2X^2\beta_X) = 0, \quad (\text{B8})$$

meaning that we require $\alpha \neq 0$ and $\alpha - X\alpha_X + 2X^2\beta_X \neq 0$, which is the case so long as the invertibility condition (2.6) is satisfied.

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